Adaptive image restoration using discrete polynomial transforms

Xavier Neyt, Marc Acheroy

Signal and Image Center, Elec. Eng. Dept., Royal Military Academy*

30 Av. de la Renaissance, B–1040 Brussels, Belgium

Abstract — **This paper presents a restoration algorithm based on a local signal description using discrete polynomials. The algorithm is made adaptive by estimating the local signal-to-noise ratio and by computing the corresponding deblurring filter. Furthermore, this method is developed for discrete signals, the input and output images being almost always available as discrete signals.**

I. Introduction

Methods to describe, restore and compress signals by mean of polynomials have already been developed by Martens [1], [3] and Philips [7]. The basic idea behind these methods is the computation of filters in order to estimate the polynomial coefficients describing the ideal signal, starting from the degraded signal.

Martens [3], applying these methods to image restoration assumes that each sample of the sampled degraded image corresponds to the zero-order term of the ideal image polynomial expansion. This implies that the blurring kernel is identical to the squared local window function used to describe the signal.

In the proposed method, no other assumption is made about the blurring kernel than a general low-pass behaviour. This allows the choice of arbitrary-shaped blurring functions and of arbitrary positions for the localisation windows.

Further, the developed method addresses the restoration of discrete signals, and more specifically of sampled images.

Section II describes briefly the polynomial transform. Section III presents its use for restoration purposes. In the same section, we will show the utility of an adaptive method, which will be described in section IV. Finally section V, presents and discusses some results.

II. Discrete polynomial transforms

This section shows how signals can be approximated using discrete polynomial transforms. The one-dimensional case will first be considered and will then be extended to two dimensional signals. Finally, this theory will be applied using discrete polynomials.

The theory exposed in this section is based on the work of Martens [1] redrawn to consider discrete functions.

A. The one dimensional discrete polynomial transform

This process includes two steps. First, local versions of the signal are produced by multiplying it by a window function $V(i)$. Second, the windowed version of the signal is approximated by a polynomial.

*The authors can be contacted via E-mail at the following addresses: Xavier.Neyt@elec.rma.ac.be and Marc.Acheroy@elec.-rma.ac.be

In the case of equally spaced windows, the following weighting function can be defined

$$W(i) = \sum_k V(i - k\Delta) \tag{1}$$

which is the periodic repetition with period Δ of the window function.

Provided $W(i)$ is non-zero for all i, the original signal can be recovered

$$L(i) = \sum_k \frac{L(i)V(i - k\Delta)}{W(i)} \tag{2}$$

Note that all the information available in the original signal is also included in the windowed versions of the signal.

The polynomial used to approximate the windowed versions of the signal is a linear combination of the basis polynomials $G_n(i)$ — where n is the degree of G_n — orthonormals with respect to the weighting function $V^2(i)$, i.e.

$$\sum_{k=-\infty}^{+\infty} V^2(k)G_m(k)G_n(k) \ = \ \delta_{nm}. \tag{3}$$

Provided the original signal $L(i)$ is square summable, one gets

$$V(i - k\Delta)L(i) = \sum_{n=0}^{\infty} L_{n,k}\, G_n(i - k\Delta)\, V(i - k\Delta) \tag{4}$$

where

$$L_{n,k} = \sum_{i=-\infty}^{+\infty} L(i)G_n(i - k\Delta)V^2(i - k\Delta). \tag{5}$$

The series in (4) converges if $L(i)$ is finite for all i. Therefore, the approximation error can be made arbitrarily small by taking the maximum degree of the polynomial expansion sufficiently high. Conversely, the original signal can be approximated with an arbitrarily small approximation error by specifying a sufficient number of coefficients.

Equation (5) expresses $L_{n,k}$ as the result of a discrete convolution subsampled by Δ

$$L_{n,k} = [L(i) * D_n(i)]_{k\Delta} \tag{6}$$

where the subscript $k\Delta$ means that the expression is evaluated at $k\Delta$ and

$$D_n(i) = G_n(-i)V^2(-i). \tag{7}$$

Finally, the direct polynomial transform consists in obtaining the set of coefficients $L_{n,k}$ where $L_{n,k}$ is the k^{th} sample of the filtered version of the original signal using a filter whose impulse response is D_n.

B. The one dimensional inverse discrete polynomial transform

The inverse discrete polynomial transform aims at reconstructing the signal from the polynomial coefficients $L_{n,k}$.

Combining equations (2) and (4), we get

$$L(i) = \sum_{k} \sum_{n=0}^{\infty} L_{n,k}\, P_n(i - k\Delta). \tag{8}$$

where

$$P_n(i) = \frac{G_n(i) V(i)}{W(i)}. \tag{9}$$

Hence, the original signal can be reconstructed by interpolating the coefficients $L_{n,k}$ with the pattern functions P_n.

C. The two dimensional discrete polynomial transform

The polynomial transform in two dimensions is a straightforward extension of the one-dimensional case.

Orthonormal polynomials are associated with the window function $V(i,j)$. These polynomials $G_{m,n-m}(i,j)$ of degree m in i and $n-m$ in j and orthonormal with respect to $V^2(i,j)$, are given by

$$\sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} V^2(i,j) G_{m,n-m}(i,j) G_{k,l-k}(i,j) = \delta_{mk}\delta_{nl}. \tag{10}$$

Equation (4) becomes

$$V(i-k\Delta, j-l\Delta) \cdot \left[L(i,j) - \sum_{n=0}^{\infty} \sum_{m=0}^{n} L_{m,n-m,k,l}\, G_{m,n-m}(i-k\Delta, j-l\Delta) \right] = 0 \tag{11}$$

where

$$L_{m,n-m,k,l} = \sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} L(i,j) \cdot G_{m,n-m}(i-k\Delta, j-l\Delta) V^2(i-k\Delta, j-l\Delta) \tag{12}$$

It follows that the coefficients are obtained by subsampling the result of the discrete convolution of the original signal $L(i,j)$ with a filter whose impulse response is given by

$$D_{m,n-m}(i,j) = G_{m,n-m}(-i,-j) V^2(-i,-j) \tag{13}$$

When the window function is separable, i.e.

$$V(i,j) = V(i)\, V(j), \tag{14}$$

the functions derived from this window function are also separable, in particular the filter impulse responses $D_{m,n-m}(i,j)$ and the pattern functions $P_{m,n-m}(i,j)$, which drastically simplifies the computations of the polynomial transforms.

D. The discrete Hermite transforms

Martens [1] suggests to select Gaussian windows and thus Hermite basis polynomials in the case of continuous signals. Similarly, [1] considers binomial window functions in the discrete case

$$V^2(i) = \begin{cases} \frac{1}{2^M} C_M^i & i = 0, ..., M \\ 0 & \text{elsewhere} \end{cases} \tag{15}$$

The associated discrete orthonormal polynomials are known as the Krawtchouk polynomials

$$G_n(i) = \frac{1}{\sqrt{C_M^n}} \sum_{k=0}^{n} (-1)^{n-k}\, C_{M-i}^{n-k}\, C_i^k. \tag{16}$$

It can be shown [8] that the central frequency of the analysis bandpass filter D_n increases with n. Hence, a limitation of the polynomial expansion order will act as a low-pass filtering of the signal.

III. Non adaptive restoration

A. Introduction

Let $L_b(i)$ be the degraded version of the ideal signal $L_o(i)$ using a blurring filter $B(i)$. If the acquisition noise $Q(i)$ is additive, $L_b(i)$ is given by

$$L_b(i) = Q(i) + B(i) * L_o(i). \tag{17}$$

This is also depicted in figure 1

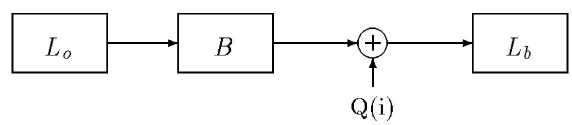

Fig. 1. Image degradation model with additive noise

The restoration algorithm consists in computing the coefficients of the polynomial expansion of the ideal signal from its degraded version $L_b(i)$.

Hence, assuming that the ideal signal can be described locally with the coefficients of a polynomial expansion and that those coefficients can be estimated from the degraded signal using filters whose impulse response is H_n, one gets the estimate of the coefficient corresponding to the polynomial of degree n in the k^{th} window by a convolution product

$$\hat{L}_{n,k} = \sum_{i} L_b(i) H_n(k\Delta - i). \tag{18}$$

Those estimates are then used to estimate the ideal signal with equation (8).

Figure 2 illustrates this restoration algorithm.

Note that, because of the noise included in the blurred signal, it is not possible to accurately estimate the high order polynomial coefficients.

Therefore it is necessary to limit the restoration procedure in order to avoid artefacts in the restored image. This limitation of the deblurring procedure is the main disadvantage of the space-invariant methods.

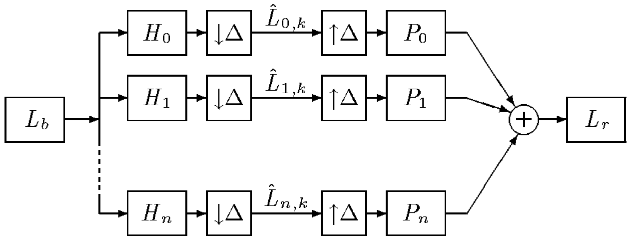

Fig. 2. Restoration algorithm

B. Determination of the filter coefficients $H_n(i)$

These filter coefficients are obtained by minimising the mean squared error (MSE) between the unknown coefficients $L_{n,k}$ and their estimates $\hat{L}_{n,k}$.

Let $L(i)$ be the ideal signal, then

$$L_{n,k} = [L(i) * D_n(i)]_{k\Delta}\,. \tag{19}$$

The MSE between this coefficient and its estimate is

$$\varepsilon^2_{n,k} = E\left((\hat{L}_{n,k} - L_{n,k})^2\right). \tag{20}$$

If the noise is assumed to be zero-mean white and uncorrelated with the ideal signal, the only noise characteristic of interest is the variance

$$E\left(Q(i)^2\right) = s_Q^2. \tag{21}$$

Further, the autocorrelation function of the ideal signal in the neighborhood of $k\Delta$ is given by

$$R_k(p,q) = E\left(L(k\Delta - p)L(k\Delta - q)\right) \tag{22}$$

it follows from (17), (18) and (20)

$$\begin{aligned}\varepsilon^2_{n,k} = \; & s_Q^2 \sum_j H_n^2(j) \\ & + \sum_{p,q} R_k(p,q)\left(\sum_i H_n(i)B(p-i) - D_n(p)\right) \\ & \cdot \left(\sum_j H_n(j)B(q-j) - D_n(q)\right)\end{aligned} \tag{23}$$

If the signal is assumed to be stationary in the neighborhood of $k\Delta$, i.e. the statistical characteristics of the signal are shift-invariant in this neighborhood, the autocorrelation becomes

$$R_k(p,q) = R_k(p-q) = R_k(q-p) \tag{24}$$

and the MSE can be rewritten

$$\begin{aligned}\varepsilon^2_{n,k} = \; & s_Q^2 \sum_j H_n^2(j) \\ & + [D_n(-p) * D_n(p) * R_k(p)]_0 \\ & - 2\sum_i H_n(i)\,[D_n(p) * B(-p) * R_k(p)]_i \\ & + \sum_{i,j} H_n(i)H_n(j)\,[B(-p) * B(p) * R_k(p)]_{i-j}\end{aligned} \tag{25}$$

At the minimum of this error, the partial derivative of $\varepsilon^2_{n,k}$ with respect to $H_n(i)$ equals zero which yields

$$\begin{aligned}\sum_j H_n(j)\left(s_Q^2\delta_{ij} + [B(-p) * B(p) * R_k(p)]_{i-j}\right) \\ = [D_n(p) * B(-p) * R_k(p)]_i\end{aligned} \tag{26}$$

Since the latter equality must be satisfied for all values of i, a system of linear equations is obtained whose solution gives the coefficients of the desired filters H_n.

In the left-hand term of eq. (26), the coefficients $H_n(j)$ are independent on the order n. It follows that, only one matrix inversion is required to solve this system.

Still in order to simplify these equations, the signal autocorrelation support is assumed to be much smaller than the size of the blurring kernel B. This is equivalent to write

$$R_k(p) = s_k^2\delta(p) + m_k^2 \tag{27}$$

where s_k^2 is the local signal variance and m_k its local mean. Even though this hypothesis seems not to be realistic for the whole image, it remains a good approximation where the deblurring will have the largest effect, i.e. where there are large intensity changes and hence a small correlation support.

To get rid of the local mean m_k in the above equation, the following normalisation condition is added

$$\sum_j H_n(j) = \frac{\sum_p D_n(p)}{\sum_p B(p)}, \tag{28}$$

which yields a simplified equation

$$\begin{aligned}\sum_j H_n(j)\left(\frac{s_Q^2}{s_k^2}\delta_{ij} + [B(p) * B(-p)]_{i-j}\right) \\ = [D_n(p) * B(-p)]_i\end{aligned} \tag{29}$$

The size of the system to be solved depends on the support size of the filter kernel H_n which in turn depends on the width of the localisation window and on the support size of the blurring kernel.

C. Local signal-to-noise ratio

The local SNR of the degraded signal is defined by

$$\mathrm{SNR}_k = 10\log\frac{s_k^2}{s_Q^2}. \tag{30}$$

This ratio plays a leading role in the computation of the filters H_n. Indeed, it determines to which extend the high spatial frequencies can be amplified.

Combining equations (25) and (26) yields the expression of the minimum MSE

$$\begin{aligned}\varepsilon^2_{n,k} = \; & [D_n(-p) * D_n(p) * R_k(p)]_0 \\ & - \sum_i H_n(i)[D_n(p) * B(-p) * R_k(p)]_i\end{aligned} \tag{31}$$

and the expectation of $L_{n,k}^2$ is

$$E(L_{n,k}^2) = [D_n(i) * D_n(-i) * R_k(i)]_0, \qquad (32)$$

The assumption of the small correlation support of the original signal yields

$$\frac{\varepsilon_{n,k}^2}{E(L_{n,k}^2)} = 1 - \frac{\sum_i H_n(i)[D_n(p) * B(-p)]_i}{[D_n(i) * D_n(-i)]_0} \qquad n \neq 0 \qquad (33)$$

which corresponds, by definition, to the inverse of the mean signal-to-noise ratio of the coefficient $L_{n,k}$.

Figure 3 presents the SNR of the coefficients as a function of the order n for different local SNR_k of the degraded signal. Note that the SNR of the coefficients de-

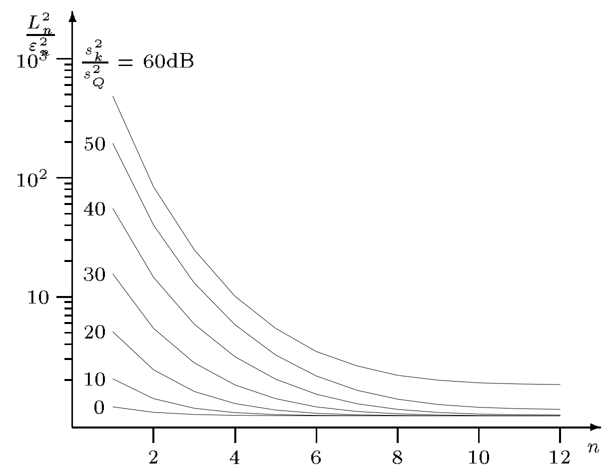

Fig. 3. Coefficient SNR as a function of the order n for different local signal-to-noise ratios

creases when the order n increases. Further, the SNR of the coefficients needs to be large enough in order to avoid artifacts. If follows that only a limited number of coefficients can be used which will limit the order of the polynomial transform.

D. *Frequency response of the restoration system*

Replacing in the reconstruction equation (8) the coefficient $L_{n,k}$ by its value estimated from equation (18) yields

$$\hat{L}_r(j) = \sum_{n=0}^{N} \sum_i L_b(i) \sum_k H_n(k\Delta - i)\, P_n(j - k\Delta). \qquad (34)$$

This equation shows that the degraded signal undergoes a space variant linear transformation, which only becomes invariant if the inter–window spacing equals unity.

The Fourier analysis is unusable to study the frequency response of this transform because it only applies to linear space invariant transformation. However, the Fourier transform of the impulse response in a given position give an evaluation of the frequency response in the neighborhood of that position. Furthermore, in practice, for a moderate inter-window distance Δ, the impulse response hardly depends on the position.

Figure 4 depicts the modulus of the Fourier transform of the impulse response in the center of a localisation window, for different values of the local SNR $\frac{s_k^2}{s_Q^2}$. Note

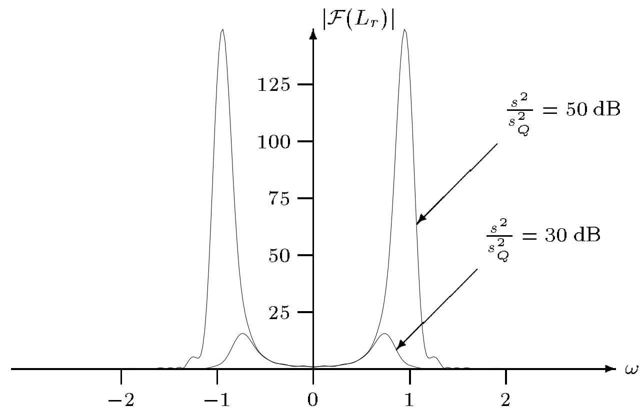

Fig. 4. Global frequency response of the deblurring system

that, as expected, the system has a global high-pass behaviour, the very high frequency being cut off. This cut off has two reasons. First, an insufficient signal-to-noise ratio of the degraded signal results in a too small SNR for the high spatial frequencies. Second, only a limited polynomial expansion has been considered.

E. *Two dimensional restoration*

The restoration of two dimensional signals can be handled in two ways. A first method consists in a simple extension of the 1D case, and the 2D filter coefficients are given by

$$\hat{L}_{m,n-m,k,l} = \sum_{i,j} L_b(i,j)\; H_{m,n-m}(k\Delta - i, l\Delta - j) \qquad (35)$$

where the filters $H_{m,n-m}$ are solution of the linear system

$$\sum_{i',j'} H_{m,n-m}(i',j') \left(\frac{s_Q^2}{s_k^2} \delta_{ii'} \delta_{jj'} + [B(p,q) * B(-p,-q)]_{i-i',j-j'} \right) = [D_{m,n-m}(p,q) * B(-p,-q)]_{i,j} \quad \forall i,j \qquad (36)$$

with

$$\sum_{i,j} H_{m,n-m}(i,j) = \frac{\sum_{p,q} D_{m,n-m}(p,q)}{\sum_{p,q} B(p,q)}. \qquad (37)$$

A second method consists in approximating the filters $H_{m,n-m}$ by the product of one-dimensional filters $H_n(i) \cdot H_{n-m}(j)$ obtained in the same way as in the one-dimensional case. Note that even if the filters $D_{m,n-m}$ and B are separable, the filters $H_{m,n-m}$ generally are not separable because of the presence of $\frac{s_Q^2}{s_k^2}$ in equation (36).

IV. Adaptive image restoration

It has been shown above that the filters $H_{m,n-m}$ depend on the local signal variance s_k^2 which must be estimated.

In the non adaptive method, a constant value has been selected for the local signal variance. Consequently, to make the algorithm adaptive, the local variance must be estimated in each window, and the corresponding filters must be computed.

A. Local variance estimation

It is easy to show that the mean energy of the estimated coefficients is given by [8]

$$E(\hat{L}_{n,k}^2) = s_Q^2 \sum H_n^2(i) + s_k^2 \sum_{i,j} H_n(i) H_n(j) \left[B(p) * B(-p)\right]_{i-j} \quad (38)$$

for $n > 0$. Considering the equation system (29) and assuming

$$\begin{aligned} A_Q(n) &= \sum_i H_n^2(i) \\ A_F(n) &= \sum_i H_n(i) \left[D_n(p) * B(-p)\right]_i \\ A_L(n) &= A_F(n) - \frac{s_Q^2}{s^2} A_Q(n) \end{aligned} \quad (39)$$

where s^2 is an initial guess of the ideal signal variance used to compute the filters H_n (see (27) and (29)), yields

$$E(\hat{L}_{n,k}^2) = s_Q^2 A_Q(n) + s_k^2 A_L(n). \quad (40)$$

Hence, adding eq. (40) up to order N, we can write

$$s_k^2 = \frac{\sum_{n=1}^{N} E(\hat{L}_{n,k}^2) - s_Q^2 \sum_{n=1}^{N} A_Q(n)}{\sum_{n=1}^{N} A_L(n)}. \quad (41)$$

One should mention that if $s^2 = s_k^2$, i.e. if the guessed variance equals the local signal variance, the above equation simply becomes

$$s_k^2 = \frac{\sum_{n=1}^{N} E(\hat{L}_{n,k}^2)}{\sum_{n=1}^{N} A_F(n)} \quad (42)$$

In practice, the noise will always be sufficiently low to neglect the term s_Q^2 in equations (39) and (41), the latter becoming identical to eq. (42).

In order to estimate the local signal variance using eq. (41), the coefficients of the polynomial expansion in that window are needed and will be computed using the filters H_n obtained by solving the system (29) selecting for the variance s_k^2 a value which minimises the error on the polynomial coefficients.

B. Mean error on the coefficient $\hat{L}_{n,k}$

In order to find the best value for s^2 in (29) to compute the filter coefficients $H_n(i)$ which are used to compute the polynomial coefficients $\hat{L}_{n,k}$, and finally to determine the local variance of the ideal signal, let us compute the mean error on the coefficient $\hat{L}_{n,k}$ when the local signal variance is different from the signal variance used to find the filter coefficients $H_n(i)$.

The expected squared estimation error on the coefficient $\hat{L}_{n,k}$, normalised with respect to the noise variance is given by [8]

$$\frac{\varepsilon_{n,k}^2}{s_Q^2} = A_Q(n) + \frac{s_k^2}{s_Q^2} A_S(n) \quad (43)$$

where A_Q has been defined above and

$$A_S(n) = \left[D_n(-p) * D_n(p)\right]_0 - A_F - \frac{s_Q^2}{s^2} A_Q \quad (44)$$

In this equation, s^2 refers to the variance used to compute the filter coefficients $H_n(i)$ which are used to evaluate the

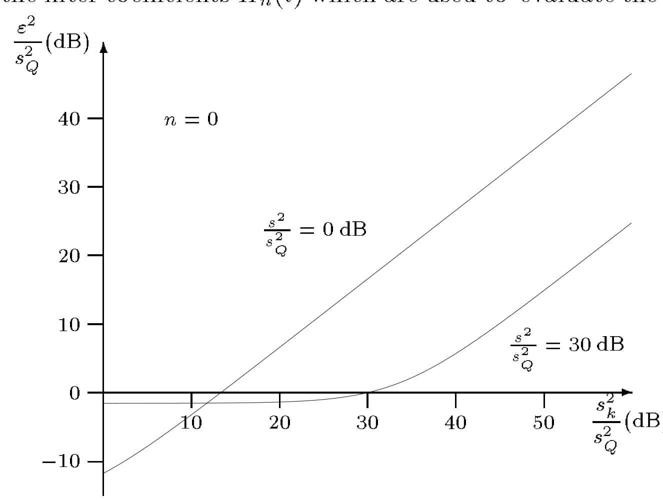

Fig. 5. Mean squared error for $n = 0$

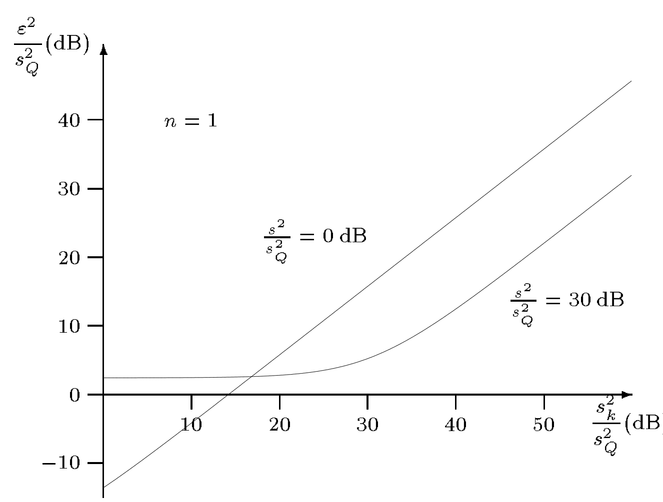

Fig. 6. Mean squared error for $n = 1$

functions A_S, A_F and A_Q, while s_k^2 is the real local signal variance.

The expected squared estimation error versus the local variance, both normalised with respect to the noise variance, has been plotted in figure 5 and 6 according to eq. (43) for two different filters. The first one is optimised for a small local SNR ($\frac{s^2}{s_Q^2} = 0$ dB) and the second for a high local SNR ($\frac{s^2}{s_Q^2} = 30$ dB).

Note on these figures that in the case of the high-SNR filter, the local signal variance must exceed a threshold in order to give better results than the low-SNR filter. For low orders, this threshold is very low. Hence, the filter optimised for high SNR will always be used to compute low-order polynomial coefficients. Conversely, for high orders, this threshold will be quite high and the low SNR filters will be used. For intermediate orders, the local signal variance must be estimated to compute the corresponding filter coefficients $H_n(i)$.

C. Adaptive restoration method

The adaptive restoration method can be divided into two steps. First, an estimate of the local signal variance will be computed in order to compute the filter coefficients $H_n(i)$ which are finally used to estimate the polynomial coefficients $\hat{L}_{n,k}$ of the ideal signal.

However, in practice, only two families of filters will be used. The one will be optimised for high-SNR signals and will be used to estimate the local signal variance, the other for low-SNR.

For this last family, the null filter can even be considered which always return zero. Indeed, these filters will only be used to estimate high order coefficients in areas of low SNR and according to figure 4, these coefficients will always be taken equal to zero.

Fig. 7. Artificially blurred image

V. Results

In this section, some results are presented and discussed. First, the deblurring algorithm is applied to an artificially blurred image, i.e. an image filtered with a known filter B. The method is then used to restore a real image, i.e. taken with an out-of-focus camera.

Obviously, the signal-to-noise ratio of the artificially blurred image will be very high. Indeed, the only remaining noise is the quantization noise, the sensor noise being filtered with the blurring.

Satisfying results are obviously obtained, even when a non adaptive approach is used, as shown in figure 8.

Fig. 8. Restoration of an artificially blurred image

The use of an adaptive method enhances slightly the restoration of the high spatial frequencies.

When real degraded images are used, the blurring filter B must be known. Figure 9 shows the considered filter, which is a mathematical model of the blur, developed in [8] and adjusted by trial and error.

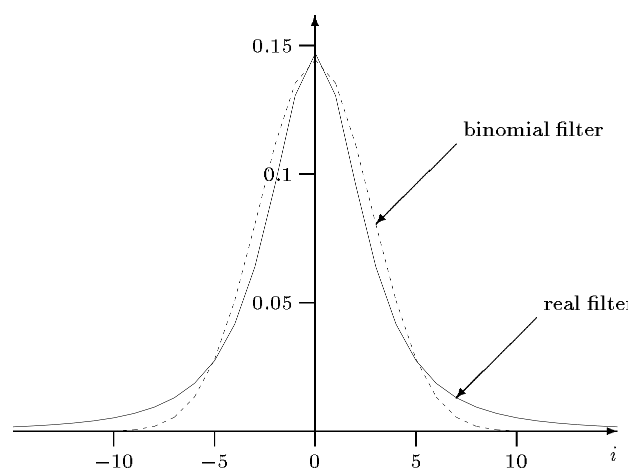

Fig. 9. Model of the real blurring filter

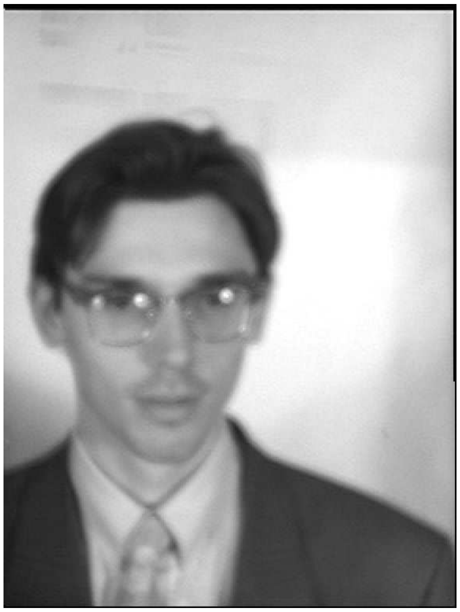

Fig. 10. Real blurred image

Another drawback of real images is their low signal-to-noise ratio. The result of a non adaptive method is shown in figure 11.

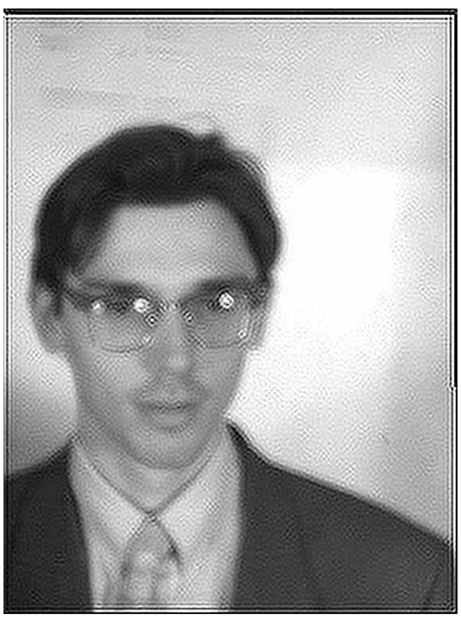

Fig. 11. Non adaptive restoration of a real image

The use of the adaptive algorithm allows to reduce the noise appearing in the uniform regions of figure 11, as shown in figure 12.

VI. Conclusions

In this paper, a deblurring method based on the use of a local polynomial approximation of the signal has been presented. Such a local description is particularly well suited for adaptive restoration methods. It has been shown how to make the algorithm adaptive with respect

Fig. 12. Adaptive restoration of a real image

to the signal-to-noise ratio but a spatially variant blurring filter B could easily be considered since no assumption has been made about the blurring filter. Note finally that this local description enables the easy parallelisation of the algorithms.

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