Dispersion of an artillery projectile due to its unbalance

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The ballistic dispersion of conventional artillery shells is due mainly to the dispersion of the initial velocity, the quadrant elevation, the angle of bearing, the projectile mass and the initial transversal angular velocity. This paper presents a method to calculate the contribution to ballistic dispersion of the initial transversal angular velocity. The latter is assumed to be the consequence of the gap between the tube and the projectile and to be due to the static and dynamic unbalance of the projectile. This method also permits to determine a maximum allowed unbalance level during fabrication of the projectile, to determine the tolerances between the tube and the projectile, and so to contribute to the prediction of the ballistic life of the tube.

Nomenclature

$\vec{V}$	aerodynamic velocity
$\vec{V}_k$	flight velocity
$\vec{V}_W$	velocity of the air, wind
$\mathbf{L}_X, \mathbf{L}_Y, \mathbf{L}_Z$	transformation matrices around x-axis, y-axis and z-axis
$\mathbf{L}_{SF}$	transformation matrix from body axes to stream axes
$\vec{a}_K$	Coriolis acceleration
$\alpha \quad \beta$	angle of attack in the plane xz, and side slip angle from the same plane
$\overline{\alpha}$	total angle of attack in the stream plane $\overline{x}\,\overline{z}$
$\delta_X \quad \delta_Y \quad \delta_Z$	angles between aerodynamic and body axes
$\vec{\Omega}$	angular velocity vector
$\psi \quad \vartheta \quad \phi$	horizontal, vertical and spinning angle of the projectile
d	caliber

Components of ... vector with respect to ... axes	Body axes (principal axes of inertia) $x \quad y \quad z$	Stream axes (geometric axes of exterior projectile surface) $\bar{x} \quad \bar{y} \quad \bar{z}$
Velocity	$u \quad v \quad w$	$\bar{u} \quad \bar{v} \quad \bar{w}$
Angular velocity	$p \quad q \quad r$	$\bar{p} \quad \bar{q} \quad \bar{r}$
Aerodynamic force coefficients	$C_X \; C_Y \; C_Z$	$C_{\bar{X}} \; C_{\bar{Y}} \; C_{\bar{Z}}$
Aerodynamic moment coefficients	$C_\ell \; C_m \; C_n$	$C_{\bar{\ell}} \; C_{\bar{m}} \; C_{\bar{n}}$
Aerodynamic force	$X \quad Y \quad Z$	$\bar{X} \quad \bar{Y} \quad \bar{Z}$
Aerodynamic moment	$L \quad M \quad N$	$\bar{L} \quad \bar{M} \quad \bar{N}$

Indexes:

$(\;)_0$ initial conditions at the muzzle

$(\;)_S$ belongs to the stream system

$(\;)_F$ belongs to the body system

$(\;)_G$ belongs to the gun system

$(\;)_O$ belongs to the local fixed coordinate system

$(\;)_E$ belongs to the earth fixed system

$(\;)^* = (\;)\dfrac{d}{V}$ dimensionless angular velocity

1 INTRODUCTION

In the basic 6DOF calculation of the motion of classical artillery shells one accepts the shells to be ideal. This means that the axis of symmetry of the exterior surface coincides with the longitudinal principal axis of inertia, and that the two transversal principal moments of inertia are identical. This hypothesis allows to project the equations of motion on the axes of the aeroballistic (non rotating) system, leading to 11 equations with 11 unknowns [1]. There is no equation with the angle of spinning of the projectile, so we avoid the problem of a small time step due to the high spinning rate. Under these hypotheses however, we can not know what happens when the symmetry axis of the exterior surface is not the principal axis of inertia.

In his analytical approach of the general motion of a projectile, Murphy [2] introduced the angle between the longitudinal axis of inertia and the aerodynamic velocity for zero lift, and later - [3] and [4] - showed the effect of a small dynamic unbalance on the angular motion. Vaughn and Wilson [5] showed that it is important for ballistic match to control the yaw of repose induced drift of a projectile. Hodapp [6] investigated the influence of the static (the center of gravity offset) and dynamic (misalignment between the principal axis of inertia and the axis of symmetry) unbalance on the trim angles K_1, K_2 and K_3. He concluded that a small mass asymmetry has proportionally large effects on the magnitude of maximum yaw and therefore unacceptable changes in range. So he proposed that it is not enough to satisfy the fundamental ballistic similitude criteria (identical shape, weight, static margin and moments of inertia) but that it is also necessary to control differences in mass asymmetry between candidate matching projectiles. Therefore Hodapp and LaFarge [7] proposed a new criterion to match the range due to mass asymmetry effect. Rollstin and Hodapp [8] confirmed experimentally by a series of flight tests the predicted decrease in range as a result of the larger angular motion caused by the imposed small principal axis misalignment.

At the moment, the possibilities of the simulation have significantly improved. In addition, using AutoCAD mass analysis, it is easy to determine the expected static and dynamic unbalance of a projectile due to the manufacturing tolerances. This paper proposes a method to compute the influence of this unbalance on the range, as well as on the dispersion of the projectile.

2 EQUATIONS OF MOTION

We accept the hypotheses that the projectile is a rigid body, the earth is flat and the wind is horizontal and not a function of the time. We neglect the angular velocity of the local fixed coordinate system $\vec{\Omega}_O$. We shall project the equations of motion on the principal axes of inertia of the projectile ($I_{XY} = I_{YZ} = I_{ZX} = 0$, $I_y \neq I_z$). With these hypotheses the equations of motion become

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} u_k \\ v_k \\ w_k \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} \dot{u}_k \\ \dot{v}_k \\ \dot{w}_k \end{bmatrix} = -\begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \begin{bmatrix} u_k \\ v_k \\ w_k \end{bmatrix} + \frac{1}{m} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \mathbf{L}_{FO} \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} - \vec{a}_K \tag{2}$$

$$\begin{bmatrix} I_x \dot{p} \\ I_y \dot{q} \\ I_z \dot{r} \end{bmatrix} = -\begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \begin{bmatrix} I_x p \\ I_y q \\ I_z r \end{bmatrix} + \begin{bmatrix} L \\ M \\ N \end{bmatrix} \tag{3}$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\vartheta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\, tg\vartheta & \cos\phi\, tg\vartheta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi/\cos\vartheta & \cos\phi/\cos\vartheta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{4}$$

The state vector for this system of differential equations is

$$[x \quad y \quad z \quad u_k \quad v_k \quad w_k \quad p \quad q \quad r \quad \phi \quad \vartheta \quad \psi]^T \tag{5}$$

3 THE UNBALANCED PROJECTILE

If a projectile is not ideal (unbalanced), the axes of the stream system and axes of the body system do not coincide. To pass from variables with respect to the axes of the stream system to variables with respect to the axes of the body system we need the transformation matrix $\mathbf{L}_{FS}$. We shall use de Sparra's angles to find this transformation matrix from the stream system S to the body system F:

- δ_Z around *z*-axis of inertia to put the plane *xz* parallel to $\bar{x}$-axis
- δ_Y around new position of the *y*-axis of inertia to put x axis parallel to $\bar{x}$-axis
- φ around the $\bar{x}$-axis to put the $\bar{z}$-axis in the position of the *z*-axis (after rotation δ_Y).

The constant angles δ_Z and δ_Y represent the **dynamic unbalance** of the missile. The third angle φ is between the $\bar{z}$-axis and the projection of the *z*-axis of inertia on the plane $\bar{z}\,\bar{y}$. This projection is a fixed direction on the body, and therefore the angle φ measures the rotation of the body with respect to the plane of flow.

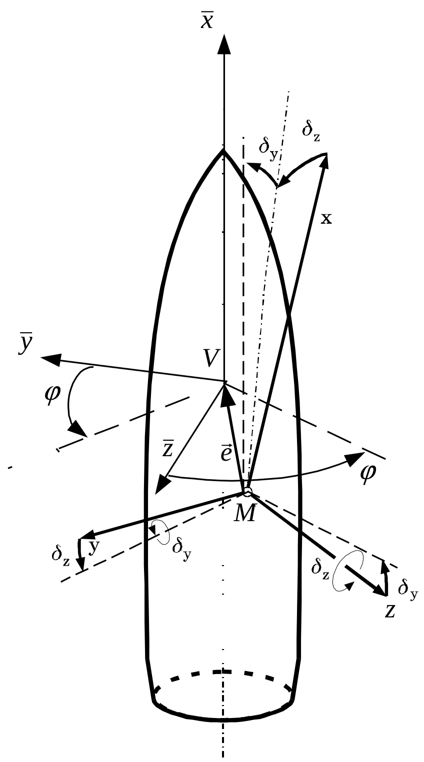

Fig.1 Position of the stream system with respect to the body system

Finally we come from the body system to the stream system by three fundamental rotations

$$\mathbf{L}_{SF} = \mathbf{L}_X(-\varphi)\mathbf{L}_Y(\delta_Y)\mathbf{L}_Z(\delta_Z)$$

The misalignment angles δ_Z and δ_Y are always very small. We can simplify the calculation by neglecting the second order small value

$$\mathbf{L}_{SF} = \begin{bmatrix} 1 & \delta_Z & -\delta_y \\ -\delta_Z \cos\varphi - \delta_Y \sin\varphi & \cos\varphi & -\sin\varphi \\ -\delta_Z \sin\varphi + \delta_Y \cos\varphi & \sin\varphi & \cos\varphi \end{bmatrix} \quad (6)$$

$\vec{e} = \overline{MV}$ is a vector from the center of mass to the reference point of the aerodynamic coefficient $\begin{bmatrix} C_{\bar{\ell}} & C_{\bar{m}} & C_{\bar{n}} \end{bmatrix}^T$. Therefore the components of this vector along the body axes are $\vec{e} = [e_X \quad e_Y \quad e_Z]^T$, where $e_X = x_V - x_M$. The other two values e_Y and e_Z are the coordinates along the body axes of the point where the axis of symmetry of the exterior surface passes through the plane of inertia M_{yz}. They represent the **static unbalance**.

4 AERODYNAMIC FORCES AND MOMENTS

Aerodynamic forces and moments are created by the air flow over the exterior surface. To find them we shall define the *stream coordinate system* (S) with respect to this surface. The origin is the aerodynamic reference point, located at a distance x_V from the nose of the projectile. The geometric axis of symmetry of the exterior surface will be the $\bar{x}$-axis of the stream coordinate system, oriented upstream. This $\bar{x}$-axis and the free stream velocity constitute the *stream plane*. The $\bar{z}$-axis is in the plane of flow upstream to the cross flow. The complete set of aerodynamic coefficients is known in this stream coordinate system [9] :

$$\begin{aligned} C_{\bar{X}} &= C_{\bar{X}0} + C_{\bar{X}\bar{\alpha}^2}\bar{\alpha}^2 & C_{\bar{\ell}} &= C_{\bar{\ell}p}\bar{p}^* \\ C_{\bar{Y}} &= C_{\bar{Y}p\bar{\alpha}}\bar{p}^*\bar{\alpha} & C_{\bar{m}} &= C_{\bar{m}\bar{\alpha}}\bar{\alpha} + C_{\bar{m}\dot{\bar{\alpha}}}\dot{\bar{\alpha}}^* + C_{\bar{m}q}\bar{q}^* \\ C_{\bar{Z}} &= C_{\bar{Z}\bar{\alpha}}\bar{\alpha} & C_{\bar{n}} &= C_{\bar{n}p\bar{\alpha}}\bar{p}^*\bar{\alpha} + C_{\bar{m}q}\bar{r}^* - C_{\bar{m}\dot{\bar{\alpha}}}\bar{\alpha}\dot{\varphi}^* \end{aligned} \quad (7)$$

The values $C_{\bar{X}0}, C_{\bar{X}\bar{\alpha}^2}, C_{\bar{Y}p\bar{\alpha}}, C_{\bar{Z}\bar{\alpha}}, C_{\bar{\ell}p}, C_{\bar{m}\bar{\alpha}}, C_{\bar{m}\dot{\bar{\alpha}}}, C_{\bar{m}q}, C_{\bar{n}p\bar{\alpha}}$ are all function of the Mach number. The aerodynamic variables, we call them the aerodynamic parameters: Mach number, $\bar{\alpha}$, $\dot{\bar{\alpha}}$, $\bar{p}$, $\bar{q}$ and $\bar{r}$, are not in the state vector, so we have to find them as a function of the state vector. We shall do that in the next paragraph.

The aerodynamic coefficients of the forces and the moments in the body system are:

$$\begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix} = \mathbf{L}_{FS} \begin{bmatrix} C_{\overline{X}} \\ C_{\overline{y}} \\ C_{\overline{z}} \end{bmatrix} \qquad \begin{bmatrix} C_\ell \\ C_m \\ C_n \end{bmatrix} = \mathbf{L}_{FS} \begin{bmatrix} C_{\overline{\ell}} \\ C_{\overline{m}} \\ C_{\overline{n}} \end{bmatrix} + \begin{bmatrix} e_X \\ e_Y \\ e_Z \end{bmatrix} \otimes \mathbf{L}_{FS} \begin{bmatrix} C_{\overline{X}} \\ C_{\overline{Y}} \\ C_{\overline{Z}} \end{bmatrix} \tag{8}$$

5 AERODYNAMIC PARAMETERS

The components of the wind are known in the carried coordinate system as a function of the height $\mathbf{V}_W^O = \begin{bmatrix} u_W^O(y) & w_W^O(y) & 0 \end{bmatrix}^T$. Therefore the components of aerodynamic velocity $\vec{V} = \vec{V}_k - \vec{V}_W$ in the body coordinate system of the projectile are

$$\begin{bmatrix} u & v & w \end{bmatrix}^T = \begin{bmatrix} u_k & v_k & w_k \end{bmatrix}^T - \mathbf{L}_{FO} \begin{bmatrix} u_W^O & w_W^O & 0 \end{bmatrix}^T \tag{9}$$

The aerodynamic velocity is

$$V = \sqrt{u^2 + v^2 + w^2} \tag{10}$$

The transformation matrix is $\mathbf{L}_{FO} = \mathbf{L}_X(\phi)\mathbf{L}_Y(\vartheta)\mathbf{L}_Z(\psi)$. The components of the aerodynamic velocity along the stream axes are (by definition of the stream axes $\overline{v} = 0$)

$$\begin{bmatrix} \overline{u} \\ 0 \\ \overline{w} \end{bmatrix} = \mathbf{L}_{SF} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u + \delta_Z v - \delta_Y w \\ u(-\delta_Z \cos\varphi - \delta_Y \sin\varphi) + v\cos\varphi - w\sin\varphi \\ u(-\delta_Z \sin\varphi + \delta_Y \cos\varphi) + v\sin\varphi + w\cos\varphi \end{bmatrix} \tag{11}$$

From the second equation of the system (11), it follows that

$$\tan\varphi = \frac{v - u\delta_Z}{w + u\delta_Y} \tag{12}$$

$$\dot{\varphi} = \frac{(\dot{v} - \dot{u}\delta_Z)(w + u\delta_Y) - (v - u\delta_Z)(\dot{w} + \dot{u}\delta_Y)}{(v - u\delta_Z)^2 + (w + u\delta_Y)^2} \tag{13}$$

The total angle of attack and its derivation are

$$\overline{\alpha} = \frac{\overline{w}}{V} \tag{14}$$

$$\dot{\overline{\alpha}} = \frac{\dot{\overline{w}} V - \overline{w}\dot{V}}{V^2} \tag{15}$$

In the equations of motion, we have the derivation of the components of the flight velocity, but here in the equations (13) and (15) we need the derivations of the components of the aerodynamic velocity. We shall get them by derivation of the previous equation (9) for the aerodynamic velocity components

$$[\dot{u} \quad \dot{v} \quad \dot{w}]^T = [\dot{u}_k \quad \dot{v}_k \quad \dot{w}_k]^T + [\vec{\Omega}][u_W \quad v_W \quad w_W]^T - \mathbf{L}_{FO}[\dot{u}_W^O \quad \dot{w}_W^O \quad 0]^T \tag{16}$$

In the equation (15) we need the derivation $\dot{\overline{w}}$ and $\dot{V}$. The first one is obtained by derivation of the third equation in the system (11), and taking in account that $\overline{v} = 0$,

$$\dot{\overline{w}} = \dot{u}(-\delta_Z \sin\varphi + \delta_Y \cos\varphi) + \dot{v}\sin\varphi + \dot{w}\cos\varphi \tag{17}$$

The easiest way to get $\dot{V}$ is to derive the equation (10). It follows that

$$\dot{V} = \frac{u\dot{u} + v\dot{v} + w\dot{w}}{V} \tag{18}$$

We also have

$$[\overline{p} \quad \overline{q} \quad \overline{r}]^T = \mathbf{L}_{SF}[p \quad q \quad r]^T \tag{19}$$

With the equations 1-19 we made the program F6DOF for simulation of the flight of an unbalanced projectile.

6 INITIAL CONDITIONS

We call the plane normal to the axis of the tube at the muzzle, *the muzzle plane*. To define the initial conditions we introduce the system *G* fixed to the gun. The origin *T* of this system is the intersection point of the tube axis with the muzzle plane. The x_G-axis is along the axis of the gun's tube and the z_G-axis is down in the vertical plane through the axis of the tube. The initial time is when the mass center of the projectile passes through the muzzle plane. Therefore at the initial time, in the muzzle plane, we have three points: the mass center *M*, the intersection of the tube axis *T*, and the intersection of the geometric axis *A* of the projectile (see figure 2).

The known muzzle velocity $\mathbf{V}_{0T}^G = [u_{0T} \quad 0 \quad 0]^T$ is the velocity of projectile's point in the *T* point. The mass center of the projectile has the velocity

$$\vec{V}_{0M} = \vec{V}_{0T} + \vec{\Omega}_0 \times \overrightarrow{TM} \tag{20}$$

The components of the initial rotation are $\Omega_0^G = [p_0^G \quad 0 \quad 0]^T$. To find the components $\vec{V}_{0M}$ along the body axes (*F*) we need the vector $\overrightarrow{TM}$, and the matrix transformation $\mathbf{L}_{FG}$.

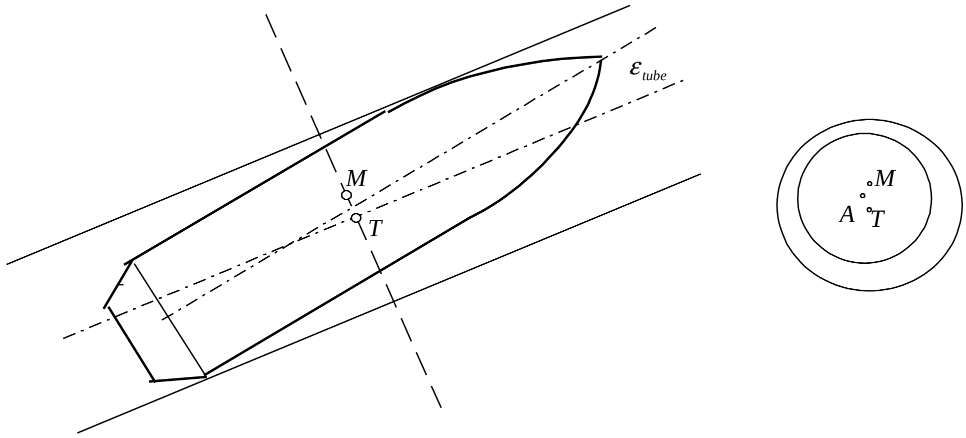

Fig.2 Projectile in the tube

The Vector $\overrightarrow{TM}$ has two parts:

- $\overrightarrow{TA}$ due to the gap between the tube and the projectile
- $\overrightarrow{AM}$ due to the unbalance of the projectile,

The components, in the body system, of the vector $\overrightarrow{\mathrm{MV}}$ are $\begin{bmatrix} x_V & e_y & e_z \end{bmatrix}^T$ therefore we shall accept that the components along the body axes of the vector $\overrightarrow{MA}$ are $\begin{bmatrix} 0 & e_y & e_z \end{bmatrix}^T$.

Let ε be the angle between the axis of the projectile and the tube. The modulus of the vector $a = \left|\overrightarrow{TA}\right|$ and ε, are geometric functions of the gap between the tube and the projectile. The components of the vector $\overrightarrow{TA}$ are $\begin{bmatrix} 0 & a\sin\varphi_0 & a\cos\varphi_0 \end{bmatrix}^T$. Therefore the components along the body axes of the resulting vector $\overrightarrow{MT}$ are (see figure 3).

$$\begin{bmatrix} 0 & e_y + a\sin\varphi_0 & e_z + a\cos\varphi_0 \end{bmatrix}^T$$

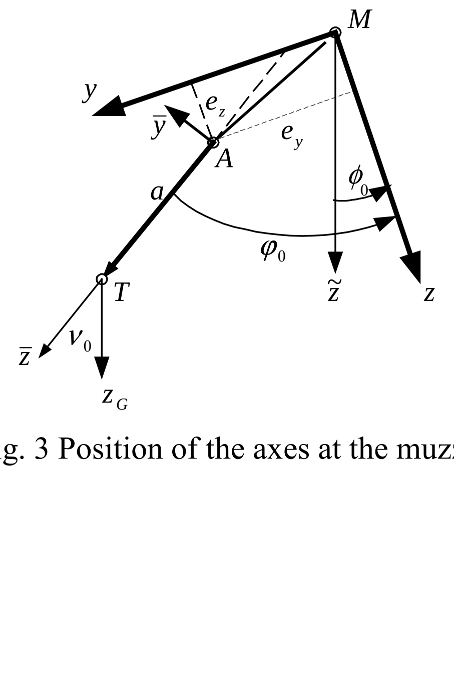

Fig. 3 Position of the axes at the muzzle

6.1 Initial conditions in the body axis system

We need the components of the initial velocity and the initial angular velocity vectors with respect to the body axes. The mass center velocity has the following components along the body axes system

$$\mathbf{V}_0 = \begin{bmatrix} u_{0T} \\ v_{0T} \\ w_{0T} \end{bmatrix} - \begin{bmatrix} 0 & -r_0 & q_0 \\ r_0 & 0 & -p_0 \\ -q_0 & p_0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ e_Y + a\sin\varphi_0 \\ e_Z + a\cos\varphi_0 \end{bmatrix} \tag{21}$$

On the right side, the components of the initial velocity vector of the point T and the components of the initial angular velocity vector, are given by the matrix equations

$$\begin{bmatrix} u_{0T} \\ v_{0T} \\ w_{0T} \end{bmatrix} = \mathbf{L}_{FG} \begin{bmatrix} u_{0T}^G \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} p_0 \\ q_0 \\ r_0 \end{bmatrix} = \mathbf{L}_{FG} \begin{bmatrix} p_0^G \\ 0 \\ 0 \end{bmatrix} \tag{22}$$

We find the matrix transformation from the G coordinate system to the body coordinate system F by passing by the stream coordinate system

$$\mathbf{L}_{FG} = \mathbf{L}_{FS} \cdot \mathbf{L}_{SG} \tag{23}$$

The first transformation matrix $\mathbf{L}_{SF}$ is given by (7). At the muzzle (see fig.3), the plane of flow, the plane of angle ε and vector $\overrightarrow{TA}$ coincide. Because the angle ε is always very small, it follows that

$$\mathbf{L}_{SG} = \mathbf{L}_Y(\varepsilon)\mathbf{L}_X(-\nu_0) = \begin{bmatrix} 1 & -\varepsilon\sin\nu_0 & -\varepsilon\cos\nu_0 \\ 0 & \cos\nu_0 & -\sin\nu_0 \\ \varepsilon_{tube} & \sin\nu_0 & \cos\nu_0 \end{bmatrix} \tag{24}$$

We can simplify the resulting matrix $\mathbf{L}_{FG} = \mathbf{L}_{FS} \cdot \mathbf{L}_{SG}$ by neglecting the second order small value. So we get finally

$$\mathbf{L}_{FG} = \begin{bmatrix} 1 & -\varepsilon\sin\nu_0 - \delta_z\cos\phi_0 - \delta_y\sin\phi_0 & -\varepsilon\cos\nu_0 - \delta_z\sin\phi_0 + \delta_y\cos\phi_0 \\ \delta_z + \varepsilon\sin\varphi_0 & \cos\phi_0 & \sin\phi_0 \\ \delta_Y + \varepsilon\cos\varphi_0 & -\sin\phi_0 & \cos\phi_0 \end{bmatrix} \tag{25}$$

With this matrix of transformation the components along the body axes of the initial angular velocity are

$$\boldsymbol{\Omega}_0 = \begin{bmatrix} p_0 \\ q_0 \\ r_0 \end{bmatrix} = \mathbf{L}_{FG} \begin{bmatrix} p_0^G \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} p_0^G \\ p_0^G (\delta_Z + \varepsilon \sin \varphi_0) \\ p_0^G (\delta_Y + \varepsilon \cos \varphi_0) \end{bmatrix} \tag{26}$$

With this angular velocity the components of the initial velocity along the body axes are

$$\mathbf{V}_0 = \begin{bmatrix} u_{0T}^G \\ u_{0T}^G (\delta_Z + \varepsilon \sin \varphi_0) \\ u_{0T}^G (\delta_Y + \varepsilon \cos \varphi_0) \end{bmatrix} - \begin{bmatrix} 0 & -r_0 & q_0 \\ r_0 & 0 & -p_0 \\ -q_0 & p_0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ e_Y + a \sin \varphi_0 \\ e_Z + a \cos \varphi_0 \end{bmatrix} \tag{27}$$

These equations give the initial velocity and the initial angular velocity as a function of the gap between the tube and the projectile, and as a function of the projectile's unbalance.

6.2 Initial condition in the ballistic coordinate system

To get the direct influence of the unbalance during the flight, we shall compare the results for an unbalanced projectile with the results for an ideal projectile, but for the same initial conditions. The results for an unbalanced projectile must be calculated with the program F6DOF, but it is better to calculate the results for an ideal projectile with the program B6DOF [1], because it is much faster.
The components of the initial angular velocity Ω_0 along the body axes are given by equation (26). The transformation matrix to the aeroballistic axes from the body axes is $\mathbf{L}_{PF} = \mathbf{L}_X(-\phi_0)$. It follows that the components of the initial angular velocity with respect to the aeroballistic axes are

$$\boldsymbol{\Omega}_{P0} = \begin{bmatrix} p_0 \\ \tilde{q}_0 \\ \tilde{r}_0 \end{bmatrix} = \mathbf{L}_{PF} \Omega_0 = \begin{bmatrix} p_0^G \\ p_0^G [\delta_Z \cos \phi_0 - \delta_Y \sin \phi_0 + \varepsilon \sin(\varphi_0 - \phi_0)] \\ p_0^G [\delta_Z \sin \phi_0 + \delta_Y \cos \phi_0 + \varepsilon \cos(\varphi_0 - \phi_0)] \end{bmatrix} \tag{28}$$

The components of the resulting vector $\overrightarrow{TM}$ with respect to the aeroballistic axis (see fig.3) are given by the matrix equation

$$\mathbf{L}_{PF} \begin{bmatrix} 0 \\ e_y \\ e_z \end{bmatrix} + \begin{bmatrix} 0 \\ a \sin \nu_0 \\ a \cos \nu_0 \end{bmatrix} = \begin{bmatrix} 0 \\ e_y \cos \phi_0 - e_z \sin \phi_0 + a \sin \nu_0 \\ e_y \sin \phi_0 + e_z \cos \phi_0 + a \cos \nu_0 \end{bmatrix} \tag{29}$$

Therefor, the initial components of the velocity along the aeroballistic axes become

$$\begin{bmatrix} u \\ \tilde{v} \\ \tilde{w} \end{bmatrix} = \mathbf{L}_{PG} \begin{bmatrix} u_{0T} \\ 0 \\ 0 \end{bmatrix} - \tilde{\Omega}_{P0} \begin{bmatrix} 0 \\ e_Y \cos\phi_0 - e_Z \sin\phi_0 + a \sin(\varphi_0 - \phi_0) \\ e_Y \sin\phi_0 + e_Z \cos\phi_0 + a \cos(\varphi_0 - \phi_0) \end{bmatrix} \tag{30}$$

In this equation the first product on the lefthand side can be calculated as follows

$$\mathbf{L}_{PF}\mathbf{L}_{FG} \begin{bmatrix} u_{0T} \\ 0 \\ 0 \end{bmatrix} = \mathbf{L}_X(-\phi_0) \begin{bmatrix} u_{0T} \\ v_{0T} \\ w_{0T} \end{bmatrix} = \begin{bmatrix} u_{0T} \\ u_{0T}\left[\delta_Z \cos\phi_0 - \delta_Y \sin\phi_0 + \varepsilon \sin(\varphi_0 - \phi_0)\right] \\ u_{0T}\left[\delta_Z \sin\phi_0 + \delta_Y \cos\phi_0 + \varepsilon \cos(\varphi_0 - \phi_0)\right] \end{bmatrix} \tag{31}$$

7 APPLICATION

7.1 Ideal projectile

We choose for an application the projectile 155 mm ER. The mechanical data for this projectile are

$$d = 0.155\,[m];\; m = 41.40\,[Kg];\; x_M = 0.693\,[m];\; I_X = 0.1376\,[Kgm^2];\; I_Z = 1.507\,[Kgm^2]$$

For all values of the Mach number we take $C_{\overline{Y}P\alpha} = 0$. The other aerodynamic coefficients are in table 1. The point of reference is $x_V = 0.579\ m$ from the top. The initial conditions without perturbations are

$$x_0 = y_0 = z_0 = 0\ \text{m}, \quad V_0 = 850\ \text{m/s} \quad p_0 = 1750\ \text{rad/s}, \quad \psi_0 = 0\ \text{rad}$$

Table 1

Ma	C_{X0}	$C_{X\alpha^2}$	$C_{\overline{Z}\alpha}$	$C_{\ell 0}$	$C_{\ell P}$	$C_{\overline{m}\alpha}$	$C_{\overline{m}\dot{\alpha}}$	$C_{\overline{m}q}$	$C_{\overline{n}0\alpha}$	$C_{\overline{n}p\alpha}$
0.20	0.1834	0.103	2.106	0.0017	-0.0466	3.223	-0.92	-1.84	-0.0120	0.138
0.40	0.1816	0.109	2.100	0.0026	-0.0520	3.265	-1.08	-2.17	-0.0120	0.138
0.60	0.1826	0.113	2.095	0.0035	-0.0570	3.332	-1.38	-2.78	-0.0120	0.138
0.80	0.1880	0.116	2.075	0.0043	-0.0617	3.468	-1.87	-3.78	-0.0120	0.138
0.95	0.2040	0.118	2.005	0.0049	-0.0650	3.737	-2.40	-4.88	-0.0120	0.138
1.00	0.2462	0.132	1.922	0.0051	-0.0661	4.042	-5.98	-12.21	-0.0129	0.144
1.05	0.3296	0.136	2.163	0.0053	-0.0671	3.780	-5.98	-12.21	-0.0138	0.149
1.10	0.3133	0.140	2.240	0.0055	-0.0681	3.714	-5.98	-12.21	-0.0147	0.155
1.15	0.3054	0.142	2.286	0.0057	-0.0691	3.654	-5.99	-12.21	-0.0156	0.161
1.20	0.2997	0.143	2.319	0.0058	-0.0701	3.612	-5.99	-12.21	-0.0165	0.167
1.30	0.2883	0.141	2.368	0.0062	-0.0720	3.552	-5.99	-12.21	-0.0183	0.178
1.60	0.2622	0.133	2.449	0.0072	-0.0772	3.421	-5.99	-12.21	-0.0193	0.185
2.00	0.2334	0.118	2.518	0.0084	-0.0830	3.287	-5.98	-12.21	-0.0206	0.194
2.50	0.2036	0.099	2.567	0.0096	-0.0884	3.118	-5.98	-12.21	-0.0206	0.194
3.00	0.1797	0.082	2.603	0.0106	-0.0918	2.945	-5.98	-12.21	-0.0206	0.194

In the case of an ideal projectile ($\delta_Y = \delta_Z = 0$, $e_Y = e_Z = 0$ and $I_Y = I_Z$) and an ideal tube ($\varepsilon = a = 0$), the two programs B6DOF from [1] and F6DOF, have to have the same results, but for appropriate steps. In the case of the trajectory $\phi_0 = 0$, $\vartheta_0 = 65^0$, we made the calculation with both programs. We compared the results for two programs with different steps of integration h. After these calculations we concluded to use the program F6DOF with the step $h = 1.0 \cdot 10^{-5}$ s, and the program B6DOF with the step $h = 1.0 \cdot 10^{-3}$ s. These are the results of both programs at the impact point:

B6DOF $t_c = 109.51$ $x_c = 21302$ $z_c = 1132$ $V_{kc} = 376.2$ $\gamma_c = -74.8$ $p_c = 846.3$

F6DOF $t_c = 109.24$ $x_c = 21300$ $z_c = 1139$ $V_{kc} = 376.2$ $\gamma_c = -74.8$ $p_c = 846.3$

7.2 Unbalanced projectile

If we have both static and dynamic unbalance, for the arbitrary chosen body axes, the dynamic unbalance $\delta_Y + i\,\delta_Z$ is a random complex variable. We know that the modulus of this complex variable $\sqrt{\delta_Y^2 + \delta_Z^2}$ follows a Rayleigh distribution with the parameter δ_m (maximum density) or δ_0 (probability = 0.5; $\delta_0 = 1.177\delta_m$). Argument of this complex value is a random variable with uniform distribution

We shall put the new z-axis of the body system along this complex variable. In this case the z-axis has the random position ϕ_0, with the uniform distribution. We have the coordinates of the impact point, range and derivation, for 8 values of initial spinning angle $\phi_0 = 0^0, 45^0, 90^0, \ldots, 315^0$, for the dynamic unbalance $\delta_0 = 0.1^0$, by both programs B6DOF and F6DOF.

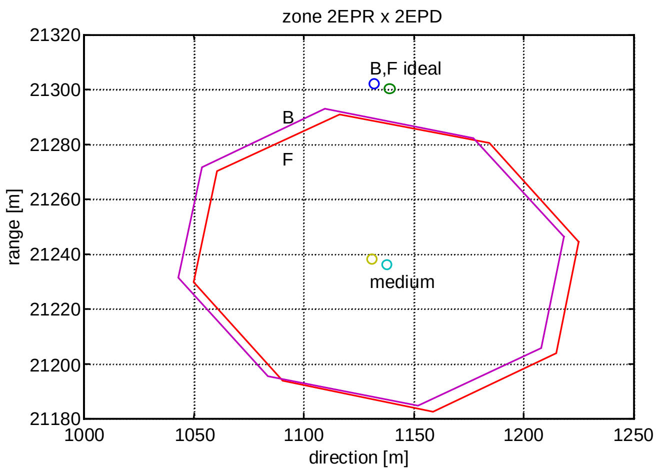

Fig.4 Miss match and dispersion due to unbalance $\delta_Z = 0.1^0$

The results are plotted on the figure 4. It is clear that the impact point with the smaller dynamic unbalance is inside the curve $X(\phi_0)$ and $Z(\phi_0)$. If we replace this curve $X(\phi_0)$ and $Z(\phi_0)$ on the fig. 4 by an ellipse along the axes of the trajectory, the radius of this ellipse along the z-axis is $(E_Z)_{dynamic}$, the probable error in direction (EPD) and the radius along the x-axis is $(E_X)_{dynamic}$, the probable error in range (EPR) due to dynamic unbalance. Distance between ideal impact point and medium impact point (see fig. 4) is the problem of "Ballistic Match" or "Ballistic similitude" which is studied in the references [3]- [8]. Here we can see both, influence of the dynamic unbalance on the ballistic match and on the dispersion of the impact point.

The same approach can be applied to calculate the range probable error and the direction probable error due to static unbalance. We did so for $e_Y = 1\,mm$, and the results are plotted on the figure 5. We see on this figure that there is not a miss match, but yet a strong influence on the dispersion. The calculations show that the probable error of dispersion is proportional to the static unbalance.

The resulting probable errors for both static and dynamic unbalances are

$$\begin{aligned}(E_X)_{unbalance} &= \sqrt{(E_X)^2_{dynamic} + (E_x)^2_{static}} \\ (E_Z)_{unbalance} &= \sqrt{(E_Z)^2_{dynamic} + (E_Z)^2_{static}}\end{aligned} \qquad (32)$$

It is possible to combine this model with the Monte-Carlo method to get the complete dispersion as it is described in [9].

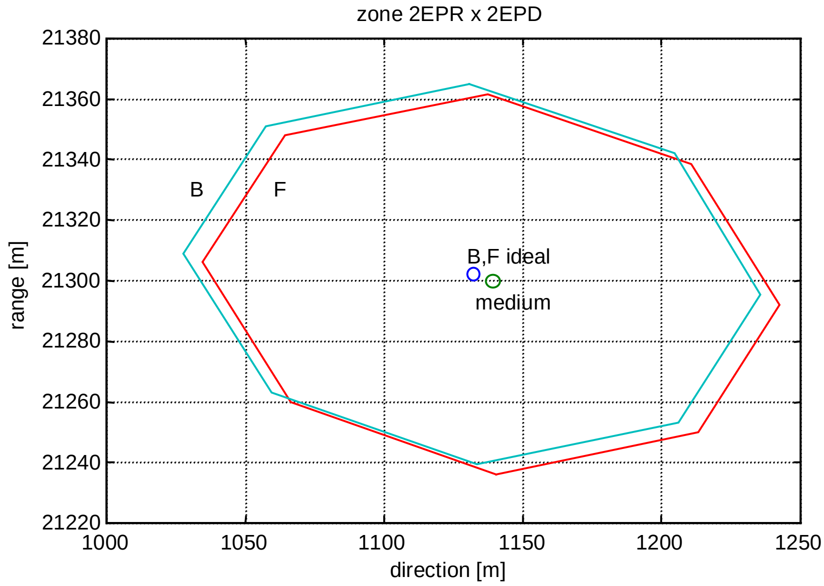

Fig. 5 $X(\phi_0)$ and $Z(\phi_0)$ for $e_Y = 1\,mm$

8 CONCLUSION

In both cases of dynamic and static unbalance we have calculated the dispersion with the program F6DOF and B6DOF. The program B6DOF does not take into account the influence of the unbalance during the flight. Nevertheless we see that the results are the same in both examples. It means that the unbalance has very small influence during the flight if it is not too big. Influence of the unbalance is due principally to the initial conditions. It allows us to make the calculations of the probable errors in range and in direction for static and dynamic unbalance by the program B6DOF, but using the adequate initial conditions for an unbalanced projectile.

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